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PLASTIC FLEXURE OF ORTHOGONALLY REINFORCED CONCRETE PLATES

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Abstract—A method of finding complete limit analysis solutions for orthotropic plates is developed. For a yield condition suitable for reinforced concrete slabs, existence of hyperbolic and parabolic stress regimes is shown. Presence of hyperbolic zones is responsible for differences between the exact and the yield line theory solutions. Differences between isotropic and orthotropic plate formulas for the yield point loads are shown. Examples of various boundary conditions are considered.

1. INTRODUCTION

THE yield line theory, its original formulation by Johansen included [1] is applicable to orthotropic plates. When applied, it furnishes an upper estimate of the collapse load. No general method existed, however, leading to complete solutions of plastic bending of such structures.

Recently, Sawczuk and Hodge [2] developed a method to establish complete solutions for isotropically reinforced concrete plates. It was shown that essentially two types of load carrying regions unfold in such plates at collapse. These regions are associated with parabolic and hyperbolic types of equations of the problem. The yield line theory accounts for parabolic regimes only. This fact produces a substantial difference in the predicted yield-point load whenever the boundary conditions impose a hyperbolic regime.

Any yield-line theory solution can, in principle, be supplemented by a lower bound estimate. For orthotropic plates, however, no general method exists for finding statically admissible stress field and the guess and trial procedures lead to quite distant bounds. It appeared therefore worthwhile to establish explicitly for orthotropic plates the relations analogous to those for isotropic ones and allowing a direct integration of stress equations.

The present note concerns this question. A suitable yield condition for reinforced concrete plates is formulated in Section 3. For orthotropic plates such a condition has to be expressed in terms of mixed tensors of moments and plastic anisotropy. The deflected surface associated with this appropriately generalized "square" yield condition is investigated in Section 4. Stress equations are studied next, yielding the relations concerning parabolic and hyperbolic regimes. The seventh section gives an analysis of stress discontinuities on the lines of change of the stress regime. Specific relations for point loaded plates are established next and differences with the isotropic case are discussed. Some examples are given in the last section. The choice of examples is such as to permit comparisons with the yield line theory solutions and to illustrate the application of the method to mixed boundary conditions as well as the presence of stress discontinuities and of various stress zones in a plate.

2. BASIC EQUATIONS

We consider an orthotropic, perfectly plastic plate. The principal axes of plastic anisotropy are chosen as Cartesian axes of reference. Let H denote the thickness, M_0 the reference yield moment per unit length, and L a reference length in the undeformed middle plane of the plate. We define dimensionless coordinates, shears, moments, curvature rates and velocity by

$$\begin{aligned} &x = X/L, \qquad q_x = LQ_x/M_0, \qquad m_x = M_x/M_0, \ldots \\ &k_x = L^2K_x/H, \ldots, \qquad w = W/H \end{aligned} \tag{2.1a}$$

while dimensionless loads by

$$p = Q/M_0 \quad \text{or} \quad PL^2/M_0 \tag{2.1b}$$

for.concentrated and distributed actions respectively. Capital letters refer to corresponding dimensioned quantities.

The dimensionless form of equilibrium equations is therefore

$$m_{x,x}+m_{xy,y} = q_x, \qquad m_{xy,x}+m_{y,y} = q_y, \tag{2.2a}$$

$$q_{x,x}+q_{y,y} = p. \tag{2.2b}$$

The curvature rates and the transverse velocity are related by

$$k_x = -w_{,xx}, \qquad k_y = -w_{,yy}, \qquad k_{xy} = -2w_{,xy}. \tag{2.3}$$

In the above, commas denote partial differentiation with respect to the ensuing variable.

In a plastic state the moments must satisfy a yield condition. For anisotropic materials such a condition necessarily involves the current tensor of plastic anisotropy, $C_{\alpha\beta\gamma\delta}$ say. Hence

$$f(C_{\alpha\beta\gamma\delta}m_{\alpha\beta}) = 0, \qquad \alpha, \beta, \gamma, \delta = 1, 2. \tag{2.4}$$

The curvature rates are then given by the associated flow rule

$$(k_x, k_y, k_{xy}) = \nu(\partial f/\partial m_x, \partial f/\partial m_y, \partial f/\partial m_{xy}), \qquad \nu \geq 0. \tag{2.5}$$

Eventually the set of equations of plastic flexure reduces to seven relations for unknown three moments, two shears, the deflection velocity w, and the scalar multiplier ν.

3. YIELD CONDITION

An element of isotropically reinforced concrete plate yields whenever the maximum principal moment attains the yield value

$$f_{\pm} = \max\{|m_1|, |m_2|\} \pm 1 = 0. \tag{3.1}$$

In a moment space m_x, m_y, m_{xy} this "square" yield condition generates a pair of intersecting coaxial cones (cf. [3], [4], [5]). For an anisotropic plate a yield criterion which appropriately generalizes (3.1) must be expressed in terms of the mixed tensor of anisotropy and moments

$$m^0_{\alpha\beta} \stackrel{\text{def}}{=} C_{\alpha\beta\gamma\delta}m_{\gamma\delta}. \tag{3.2}$$

If $m^0_{\alpha\beta}$ and $m_{\alpha\beta}$ are required to be symmetric the plastic anisotropy tensor must be of fourth order and $C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\gamma\delta\alpha\beta}$.

In the system of reference coinciding with the directions of orthogonally disposed reinforcement the matrix $C_{\alpha\beta\gamma\delta}$ is of diagonal form. This is equivalent to the statement that the yield moment for a principal direction of orthotropy depends solely on the reinforcement in this direction. Hence

$$m^0_{\alpha\beta} = C_{\alpha\beta\gamma\delta} m_{\gamma\delta} \rightarrow \begin{Vmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{Vmatrix} \cdot \begin{Vmatrix} m_x \\ m_y \\ m_{xy} \end{Vmatrix} \tag{3.3}$$

and a generalization of (3.1) is

$$f\pm = \max\{|m^0_1|, |m^0_2|\} \pm 1 = 0. \tag{3.4}$$

From (3.3) in what follows the yield moment in x-direction is taken as a reference value and $C_{\alpha\beta\gamma\delta}$ is dimensionless. In the space of "reduced" moments m^0_x, m^0_y, m^0_{xy} the condition (3.4) gives two cones

$$f_\pm = (m^0_{xy})^2 - (m^0_x \mp 1)(m^0_y \mp 1) = 0, \qquad 0 \leq \pm(m^0_x + m^0_y) \leq 2 \tag{3.5}$$

whereas in the moment space

$$f_\pm = B^2 m^2_{xy} - (m_x \mp 1)(A m_y \mp 1) = 0. \tag{3.6}$$

The upper or lower sign is to be used consistently in all equations.

The material constants A and B depend on the amount of reinforcement (here: equally disposed at the top and bottom surfaces), as well as on the concrete strength. The form (3.6) with $B^2 = A$ was obtained by Massonnet and Save [3] and Nielsen [6] through direct considerations of response of reinforced concrete slabs to bending. Moreover, Morely [7], employing bounding procedures concludes that such a particular form of (3.6) is acceptable. (cf. also Kemp [8], Save [5]). The formulation (3.4) of this yield criterion permits a direct integration of the field equations.

The yield surface (3.5) consists of two cones in the reduced stress space and intersecting the plane $m^0_{xy} = 0$ on a square. In the space m_x, m_y, m_{xy} this intersection is a rectangle (cf. [3], [7] for visualization). On intersection of the cones

$$B^2 m^2_{xy} + m^2_x = 1, \qquad m_x = -A m_y, \tag{3.7}$$

but the associated flow law contains now two scalar multipliers thus the number of equations is still equal to that of unknowns.

According to (3.5) the absolutely larger principal "reduced" moment is $m^0_1 = \pm 1$. Let m_0 denote the other principal moment and ϕ the angle between the positive x-axis and the direction associated with m_0. Equation (3.5) is then identically satisfied by the substitutions

$$\left.\begin{aligned} m^0_x &= (m_0 \mp 1)\cos^2\phi \pm 1, \qquad m^0_y = (m_0 \mp 1)\sin^2\phi \pm 1, \\ m^0_{xy} &= (m_0 \mp 1)\sin\phi\cos\phi \end{aligned}\right\}. \tag{3.8}$$

Or, restricting the analysis to the case $B = \sqrt{A}$

$$\begin{aligned} m_x \mp 1 &= \lambda \cos^2 \phi, \qquad Am_y \mp 1 = \lambda \sin^2 \phi, \\ m_{xy}\sqrt{A} &= \lambda \sin \phi \cos \phi, \qquad \lambda = (m_0 \mp 1) \end{aligned} \tag{3.9}$$

where, as above,

$$\operatorname{tg} 2\phi \equiv \frac{2m_{xy}^0}{m_x^0 - m_y^0} = \frac{2m_{xy}\sqrt{A}}{m_x - Am_y}. \tag{3.10}$$

4. DEFLECTED SURFACE

It has been remarked in [2] that a study of kinematics yields a significant information regarding integration of stress equations for isotropic plates. In view of the analogy between (3.1) and (3.4) the situation should be similar for orthotropic plates.

(*a*) *Regular moment points*

For moment points on the surface f_+ or f_-, which are away from the crease or vertices, the flow law (2.5) and the relations (2.3) produce

$$\begin{aligned} w_{,xx} &= \nu(Am_y \mp 1) = \nu\lambda \sin^2 \phi, \\ w_{,yy} &= \nu(m_x \mp 1) = \nu\lambda A \cos^2 \phi, \\ w_{,xy} &= -\nu A m_{xy} = -\nu\lambda\sqrt{(A)} \sin \phi \cos \phi. \end{aligned} \tag{4.1}$$

Equations (4.1) are a set of relations containing three unknowns w, $\nu\lambda$ and ϕ and may be discussed independently of the equilibrium equations. The type of deflected surface can be established from the sign of the discriminant of the second fundamental form of $w(x, y)$, Substitution of (4.1) into the discriminant results in

$$\Delta \equiv w_{,xx}w_{,yy} - w_{,xy}^2 = 0. \tag{4.2}$$

Hence the surface is developable, the system (4.1) is parabolic and the characteristics are straight lines. We note that for the general form of (3.6) when $B \neq \sqrt{A}$, the deflected surface is either hyperbolic or elliptic.

To obtain an explicit form of the characteristics we eliminate w among the derivatives of (4.1) and remark that further $\lambda\nu$ and its derivatives can be eliminated, rendering one quasi-linear equation in ϕ alone. Supplementing this by the expression for the variation of ϕ along the characteristic we obtain the following system

$$\begin{aligned} \sqrt{(A)}\phi_{,x} \cos \phi + \phi_{,y} \sin \phi &= 0 \\ \phi_{,x}\, \mathrm{d}x + \phi_{,y}\, \mathrm{d}y &= \mathrm{d}\phi. \end{aligned} \tag{4.3}$$

A simple computation yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A}} \operatorname{tg} \phi, \qquad \phi = \text{const.} \tag{4.4a,b}$$

Thus the characteristics (4.4) are straight and constitute one family of principal curvature lines (cf. Fig. 1). Contrary to the isotropic case, however, they are no longer the principal moment trajectories (cf. Save [5]).

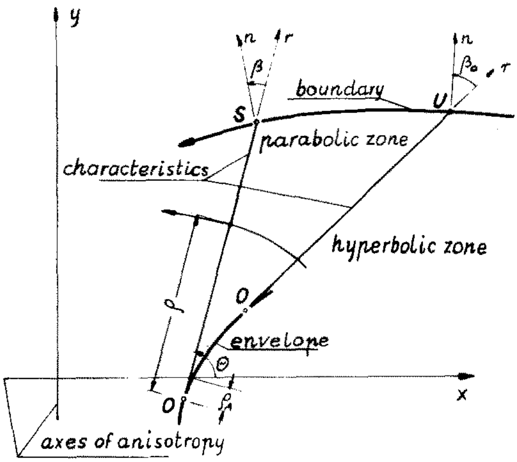

FIG. 1. Characteristics of the deflected surface.

(b) Singular moment points

On the intersection of the two cones (3.8) the stresses satisfy (3.7). Taking the first principal moment to be positive, thus

$$m_1^0 = 1, \qquad m_2^0 = -1 \tag{4.5}$$

we obtain from (3.9)

$$m_x = -\cos 2\phi, \qquad Am_y = \cos 2\phi, \qquad \sqrt{(A)}m_{xy} = -\sin 2\phi. \tag{4.6}$$

The plastic potential flow law associated with (4.5) gives

$$\begin{aligned} w_{,xx} &= (v_+ + v_-)\cos 2\phi + (v_+ - v_-), \qquad w_{xy} = \sqrt{(A)}(v_+ + v_-)\sin 2\phi \\ w_{,yy} &= -(v_+ + v_-)\cos 2\phi + A(v_+ - v_-), \qquad v_+ \geq 0, \qquad v_- \geq 0. \end{aligned} \tag{4.7}$$

The discriminant of $w(x, y)$ is easily found to be always negative

$$\Delta = w_{,xx}w_{,yy} - w_{,xy}^2 = -4Av_+v_- < 0 \tag{4.8}$$

thus the surface is of negative Gaussian curvature. From (4.7) one obtains the following equation

$$Aw_{,xx} - 2\sqrt{(A)}w_{,xy}\text{ctg}\, 2\phi - w_{,yy} = 0 \tag{4.9}$$

which is easily shown to have the characteristics

$$\frac{dy}{dx} = \frac{1}{\sqrt{A}}\text{tg}\,\phi, \qquad \frac{dy}{dx} = -\frac{1}{\sqrt{A}}\text{ctg}\,\phi. \tag{4.10a,b}$$

The net of characteristics is orthogonal only at $A = 1$, thus for the isotropic case studied in [2].

Since (4.10) contains the characteristic (4.4a), the hyperbolic and parabolic regimes juxtapose.

In the stress regions associated with the vertices of the cones the velocity field is arbitrary but of positive Gaussian curvature and subject to the restriction that the curvature-rate vector lie within the cone delimited by the normals to the yield surface at the vertices.

5. STRESS EQUATIONS: PARABOLIC ZONE

For regular points of the yield surface (3.5) the moments (3.9) when combined with the equilibrium relations (2.2a) produce the following two quasi-linear equations in ϕ and λ

$$\phi_{,x}\cos\phi+\frac{1}{\sqrt{A}}\phi_{,y}\sin\phi = \frac{1}{\lambda}(\sqrt{(A)}q_y\cos\phi-q_x\sin\phi), \tag{5.1a}$$

$$\lambda_{,x}\cos\phi+\frac{1}{\sqrt{A}}\lambda_{,y}\sin\phi = f(q_x, q_y, \lambda, \phi, \phi_{,x}, \phi_{,y}). \tag{5.1b}$$

As a simple comparison with (4.3) shows, the equations are parabolic, with the same characteristic (4.4).

Along the straight characteristics therefore

$$q_x\sin\phi-\sqrt{(A)}q_y\cos\phi = 0. \tag{5.2}$$

This, together with (2.2b) specifies the shear on the characteristic and eventually allows the integration of $d\lambda = f(\lambda, p, \phi)$. The integral takes a form particularly suitable for further interpretation if the stress equations are transformed into an orthogonal coordinate system containing the straight characteristics as one coordinate and possessing an envelope (Fig. 2). Contrary to the isotropic case this system does not constitute the net of moment trajectories. The equilibrium equations (2.2) take the form

$$rm_{r,r}+m_{r\theta,\theta}+(m_r-m_\theta) = rq_r, \tag{5.3a}$$

$$rm_{r\theta,r}+m_{\theta,\theta}+2m_{r\theta} = rq_\theta, \tag{5.3b}$$

$$rq_{r,r}+q_{\theta,\theta}+q_r+rp(\theta, r) = 0. \tag{5.3c}$$

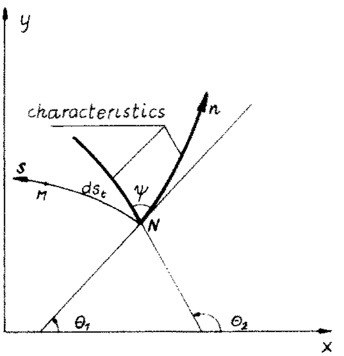

FIG. 2. Characteristics in a hyperbolic zone.

The moments (3.9) transformed to the actual system of reference for which

$$\operatorname{tg}\theta = \frac{1}{\sqrt{A}}\operatorname{tg}\phi \tag{5.4}$$

give eventually

$$\left.\begin{aligned} m_\theta &= m_x \sin^2\theta + m_y \cos^2\theta - m_{xy}\sin 2\theta = \pm\frac{1}{A}(A\sin^2\theta + \cos^2\theta) \\ m_{r\theta} &= (m_y - m_x)\sin 2\theta + m_{xy}\cos 2\theta = \pm\frac{1}{A}(1-A)\sin\theta\cos\theta \end{aligned}\right\}. \tag{5.5}$$

In view of (5.5) the equation (5.3) can be regarded as the system for the unknowns m_r, q_r and q_θ. From (5.3b,c) it is found that

$$q_\theta = 0, \qquad rq_r = C(\theta) - \int pr\,\mathrm{d}r. \tag{5.6}$$

Hence the moment along the characteristic is

$$m_r = \pm 1 \pm \frac{1-A}{A}\sin^2\theta + C(\theta) + \frac{D(\theta)}{r} + \frac{1}{r}\int pr^2\,\mathrm{d}r - \int pr\,\mathrm{d}r. \tag{5.7}$$

For an isotropic case $A = 1$, and the result of Schumann is obtained, [9].

To determine the integration function we have on a simply supported edge $m_n = 0$, thus the radial moment takes on the boundary the following value

$$\pm m_r = m_\theta \operatorname{tg}^2\beta + 2m_{r\theta}\operatorname{tg}\beta \tag{5.8}$$

where β is the angle between the boundary normal and the straight characteristic, positive when counterclockwise, (Fig. 1). To specify the stress field completely, an additional condition regarding shears is necessary. In this respect there is no difference in comparison with the isotropic case. It is worthwhile to mention that the "Affinity theorems" of the limit analysis theory (cf. [1], [10], [13]) apply in a parabolic zone of an orthotropic plate.

6. STRESS EQUATIONS. HYPERBOLIC ZONE

The moments (4.6) are easily shown to lead to a hyperbolic equation for ϕ, with (4.10) as the characteristics. To complete the solution one can transform the equilibrium equations to the system of characteristics as coordinate lines. The moments being known the equilibrium equations after elimination of shears lead to one relation between the curvatures of the coordinate net. The second relation is furnished by the conditions (4.10), specifying the angle of intersection of the coordinate lines.

Alternatively we can choose for the unknown reference system an orthogonal one, containing one of the characteristics, the first of (4.10) say, as a coordinate line (Fig. 2). Along this line

$$m_n = \frac{(A-1)^2\sin^2\theta\cos^2\theta - A}{A(A\sin^2\theta + \cos^2\theta)} = \frac{Am_{nt}^2 - 1}{Am_t}, \tag{6.1}$$

the remaining moments m_t, m_{nt} being expressed by the formulas (5.5) for m_θ and $m_{r\theta}$ respectively. The shear equilibrium equation and the Lamé condition constitute the system to be resolved for curvatures ρ_n, ρ_t. The procedure is thus analogous to that explained in [2], except that characteristics are no longer orthogonal and intersect at a variable angle ψ.

To compute the load carrying capacity Q of an hyperbolic region the "radial" shear q_n is to be integrated along a "circumferential" line

$$Q = \int_{s_0}^{s} (-q_n)\, \mathrm{d}s_t . \tag{6.2}$$

In the elliptic zone associated with the vertices of the yield surface

$$m_1^0 = m_2^0 \equiv \pm 1 \rightarrow m_x = \pm 1, \qquad Am_y = \pm 1, \qquad m_{xy} = 0, \tag{6.3}$$

as it follows from (3.8). Therefore

$$q_x = q_y = 0 \rightarrow Q = 0, \tag{6.4}$$

thus no load is carried by such a zone.

7. STRESS DISCONTINUITIES

Since the various stress regimes give different analytic solutions, stress discontinuities must be allowed on the characteristics connecting different zones. Moreover, as in a parabolic region the radial moment (5.8) depends explicitly on the boundary normal, a discontinuity can appear whenever the normal is not uniquely specified, thus at the corners. Since in both cases the connecting characteristic is straight we consider only the discontinuities allowed by the equilibrium equations in the form (5.3).

On a discontinuity line the following continuity requirements are prescribed (cf. Hopkins [11]).

$$[m_{r\theta}] = 0, \qquad [m_\theta] = 0, \qquad [q_\theta + m_{r\theta,r}] = 0. \tag{7.1}$$

We note from (5.5), (6.1) and (6.3) that neither m_θ nor $m_{r\theta}$ depend on the radial coordinate for any of the stress regimes. Assuming, for definiteness m_θ positive and employing the above stated conclusions in (5.3) we obtain the following jump conditions.

$$[q_{\theta,\theta}] \equiv 0, \qquad r[q_r] = \text{const.} \qquad [(rm_r),_r] = \text{const.} \tag{7.2}$$

Whenever $m_r = \text{const.}$ on the characteristic, it is

$$[m_r] = r[q_r] = \text{const.} \tag{7.3}$$

and the specific value of the constant is easily found to be

$$[m_r] = -m_\theta(\mathrm{tg}^2\,\beta_+ - \mathrm{tg}^2\,\beta_-) - 2m_{r\theta}(\mathrm{tg}\,\beta_+ - \mathrm{tg}\,\beta_-) \tag{7.4}$$

$$[m_r] = -m_\theta\,\mathrm{tg}^2\,\beta_+ - 2m_{r\theta}\,\mathrm{tg}\,\beta_+ - \frac{Am_{r\theta}^2 - 1}{Am_\theta} \tag{7.5}$$

for a parabolic–parabolic and a parabolic–hyperbolic boundary respectively. In (7.4), (7.5) $\mathrm{tg}\,\beta_+$ and $\mathrm{tg}\beta_-$ denote the normals to the simply supported boundary, respectively after and before passing the discontinuity line at the point where it intersects the boundary. Analogous relations are straight-forward to establish for a transition from the load carrying regions to the zone of uniform bending (6.3), where $Am_r = A\cos^2\theta+\sin^2\theta$.

It has to be remarked that for a smooth boundary, transition from a hyperbolic to a parabolic zone is continuous. This continuity requirement for m_r allows one to establish the limiting angle β_0 at which the change of stress regime occurs on a simply supported boundary. Making (5.8) equal to (6.1) one obtains eventually

$$\operatorname{tg}\beta_0 = \frac{\sqrt{A}-(1-A)\sin\theta\cos\theta}{A\sin^2\theta+\cos^2\theta}, \qquad 0 \le \theta \le \pi/2. \tag{7.6}$$

The parabolic solution applies for $0 \le |\beta| \le \beta_0$ and for $|\beta| \ge \beta_0$ the stress region is hyperbolic. If $A = 1$ the known result $\beta_0 = \pi/4$ for isotropic plates follows.

The stress equations can be used for estimations of lower bounds.

8. POINT LOADED PLATES

We proceed to establish the formulas for the limit load, in order to allow comparisons of the orthotropic plate solutions with those rendered by the yield line theory.

Let us consider a simply supported plate subjected to a single point load p applied downwards at the origin of a polar coordinate system r, θ (Fig. 3). The plate boundary is $r = \rho(\theta)$ (dimensionless). We assume that the parabolic solution will hold at least for a part of the plate. The deflected surface is over there of conical shape with vertex at the point of loading. The characteristics will then be straight lines through the origin. They form an angle β with the normal to the boundary (Fig. 3)

$$\operatorname{tg}\beta = \operatorname{tg}(\theta-\phi) = \frac{\rho'}{\rho}. \tag{8.1}$$

Since the circumferential curvature is positive, the upper sign is appropriate in Section 5. The function $D(\theta)$ in (5.7) must vanish because m_r has to be finite also at the origin. In view of the boundary condition (5.8) the moments are:

$$m_\theta = 1+(\varkappa-1)\cos^2\theta, \qquad m_{r\theta} = (\varkappa-1)\sin\theta\cos\theta, \tag{8.2a}$$

$$m_r = -\left(\frac{\rho'}{\rho}\right)^2-(\varkappa-1)\left[\left(\frac{\rho'}{\rho}\right)^2\cos^2\theta\pm\left(\frac{\rho'}{\rho}\right)\sin 2\theta\right] \tag{8.2b}$$

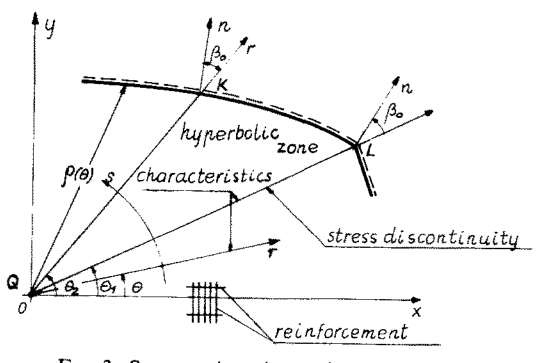

FIG. 3. Stress regimes in a point loaded plate.

where the minus sign applies whenever $\beta < 0$. To allow comparisons with the yield line theory the "orthotropy ratio" is introduced

$$\varkappa = A^{-1} \tag{8.3}$$

Substitution of these results in (5.3) yields

$$q_\theta = 0, \qquad -rq_r = 1+\left(\frac{\rho'}{\rho}\right)^2+(\varkappa-1)\left[\sin\theta\pm\left(\frac{\rho'}{\rho}\right)\cos\theta\right]^2. \tag{8.4}$$

The range of applicability of (8.1) and (8.4) is to be established from (7.6), which in view of (8.1) takes the form

$$\left(\frac{\rho'}{\rho}\right)(\sin^2\theta_0+\varkappa\cos^2\theta_0)\pm(\varkappa-1)\sin\theta_0\cos\theta_0-\sqrt{\varkappa} = 0. \tag{8.5}$$

For each particular boundary curve this has to be resolved for θ_0 delimiting the parabolic region.

The load carried by a parabolic zone is obtained by integrating the radial shear along any circumferential trajectory of the deflected surface, (Fig. 3)

$$p_p = \int_\theta^{\theta_1}(-q_r)r\mathrm{d}\theta = \int_\theta^{\theta_1}\left[1+\left(\frac{\rho'}{\rho}\right)^2+(\varkappa-1)(\sin\theta\pm\left(\frac{\rho'}{\rho}\right)\cos\theta)^2\right]\mathrm{d}\theta. \tag{8.6}$$

At $\varkappa = 1$ the known result for isotropic plates is recovered ([12], [2]).

For θ satisfying (8.5) a hyperbolic zone adjoints the parabolic zone along a straight characteristic. Such a zone is, as a rule in simply supported planes, bordered on both sides by parabolic regions. In the actually chosen system of coordinates (Fig. 3) the moments m_θ, $m_{r\theta}$ in the hyperbolic zone are given by (8.2a) and

$$m_r = \frac{(\varkappa-1)^2\sin^2\theta\cos^2\theta-\varkappa}{1-(\varkappa-1)\cos^2\theta}. \tag{8.7}$$

The shears follow from (5.3)

$$rq_r = -2\varkappa/(\sin^2\theta+\varkappa\cos^2\theta), \qquad q_\theta = 0 \tag{8.8}$$

and the load carried by the hyperbolic zone OKL in Fig. 3 is obtained from (6.2)

$$p_h = \int_{\theta_2}^{\theta_2} -rq_r\,\mathrm{d}\theta = 2\sqrt{\varkappa}\left[\mathrm{tg}^{-1}\left(\frac{\mathrm{tg}\,\theta_2}{\sqrt{\varkappa}}\right)-\mathrm{tg}^{-1}\left(\frac{\mathrm{tg}\,\theta_1}{\sqrt{\varkappa}}\right)\right]. \tag{8.9}$$

The total collapse load is obtained summing all contributions (8.6) and (8.9), since no load is carried by the zones of homogeneous bending (6.3).

When the plate boundary is clamped, the adjoining zone is hyperbolic and the collapse load consists of contributions (8.9). The situation is analogous to that in isotropic plates [2].

9. EXAMPLES

(a) *Simply supported circular plate, loaded at the center*

The directions of reinforcing fibers, and thus the axes of orthotropy are as indicated in Fig. 4. The yield moments in the directions of orthotropy are

$$m_x = \pm 1, \qquad m_y = \pm\varkappa, \qquad \varkappa \geq 1. \tag{9.1}$$

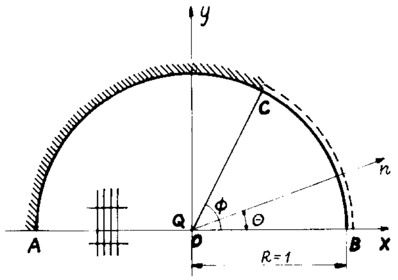

FIG. 4. Circular orthotropic plates: reinforcement arrangement and boundary conditions.

Since $\rho(\theta) = r =$ const., it is $\beta = 0$ in (8.1) and the entire plate is a parabolic stress regime. The moments are, therefore

$$m_r = 0, \qquad m_\theta = 1-(\varkappa-1)\cos^2\theta, \qquad m_{r\theta} = (\varkappa-1)\sin\theta\cos\theta \tag{9.2}$$

and the collapse load follows from (8.6)

$$p = 4\int_0^{\pi/2} [1+(\varkappa-1)\sin^2\theta]\,d\theta = (\varkappa+1)\pi, \tag{9.3}$$

being equal to the yield line theory solution. Since the stress regime is parabolic in the entire plate such a result should be expected. The yield line theory, in effect, employs the parabolic type of relations throughout.

(b) *Point loaded clamped plate*

For definiteness we take the boundary curve to be concave toward the point of loading. Because the boundary is clamped the entire plate is in a hyperbolic regime.

The moments are directly given by (8.2a) and (8.7), whereas the shears are specified by (8.8). The yield point load is therefore

$$p = 4\int_0^{\pi/2} (-q_r)r\,d\theta = 4\pi\sqrt{\varkappa}. \tag{9.4}$$

The deforming zone covers only a part of the plate.

(c) *Circular plate with mixed boundary conditions, lower bound solution*

Let the part AC of the plate shown in Fig. 4 be clamped. The zone AOC is in a hyperbolic stress regime. On the characteristic OC there is the radial moment discontinuity of magnitude

$$[m_r] = m_r^h - m_r^p = \frac{(\varkappa-1)^2\sin^2\theta\cos^2\theta-\varkappa}{1-(\varkappa-1)\cos^2\theta}, \tag{9.5}$$

since in BOC the moments, m^p, are given by (9.2) and in AOC, m^h, by (8.7) and (5.5).

The collapse load is obtained by adding the contributions of both regions. Simple computation yields

$$p = 2\sqrt{\varkappa}\left[2\pi-2\mathrm{tg}^{-1}\left(\frac{\mathrm{tg}\,\phi}{\sqrt{\varkappa}}\right)\right]+(1+\varkappa)\left(\frac{\pi}{2}-\phi\right)-\frac{(\varkappa-1)}{2}\sin\phi \tag{9.6}$$

for $0 \le \phi \le \pi/2$.

(d) *Simply-supported rectangular plate loaded at the center*

Let $2A$ and $2B$ be the sides, B the characteristic length and $\alpha = A/B \geq 1$, Fig. 5. The boundary curve is

$$r = \rho(\theta) = \alpha/\cos\theta, \qquad 0 \leq \theta \leq \theta_1,$$
$$r = 1/\sin\theta, \qquad \theta_1 \leq \theta \leq \pi/2,$$

where

$$\theta_1 = \text{ctg}^{-1}\,\alpha. \tag{9.7}$$

The orthotropy is defined by (9.1), $\varkappa > 1$.

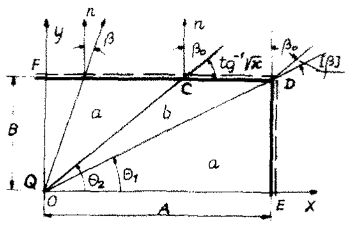

FIG. 5. Stress regimes in a rectangular plate: (a) parabolic, (b) hyperbolic.

To establish the transition line from a parabolic to a hyperbolic stress regime equation (7.6) is to be employed for positive β

$$\text{tg}\,\beta_0 \equiv \text{ctg}\,\theta_2 = \frac{\sqrt{\varkappa}-(\varkappa-1)\sin\theta_2\cos\theta_2}{\sin^2\theta_2+\varkappa\cos^2\theta_2}. \tag{9.8}$$

It yields

$$\theta_2 = \text{tg}^{-1}\sqrt{\varkappa} \tag{9.9}$$

and therefore the hyperbolic zone covers the range $\text{ctg}^{-1}\,\alpha \leq \theta \leq \text{tg}^{-1}\sqrt{\varkappa}$. The diagonal OD is a discontinuity line for the radial shear and moment.

In the parabolic zones the moments and the radial shear are given by (8.2a) and by the following expressions

$$m_r = -\sin^2\theta\,.\,(2-\varkappa+\text{tg}^2\,\theta), \qquad -rq_r = 1/\cos^2\theta \tag{9.10}$$

for $0 \leq \theta \leq \theta_1, \beta < 0$, whereas for $\theta_2 \leq \theta \leq \pi/2$, since $\beta > 0$, one obtains

$$m_r = -\cos^2\theta\,.\,(2\varkappa-1+\varkappa\,\text{ctg}^2\,\theta), \qquad -rq_r = \varkappa/\sin^2\theta. \tag{9.11}$$

In the hyperbolic zone details of the solution must be found numerically but the load carrying capacity is known from (8.9),

$$p_h = 2\sqrt{\varkappa}(\pi/4-\text{ctg}^{-1}\,\alpha\sqrt{\varkappa}). \tag{9.12}$$

Adding to (9.12) the contributions of the parabolic stress regimes the collapse load is found

$$p/4 = p_h + \int_0^{\theta_1} \frac{d\theta}{\cos^2\theta} + \varkappa \int_{\theta_2}^{\pi/2} \frac{d\theta}{\sin^2\theta}$$
$$= \alpha^{-1} + \sqrt{\varkappa}(1+\pi/2) - 2\sqrt{\varkappa}\,\mathrm{ctg}^{-1}\,\alpha\sqrt{\varkappa}. \quad (9.13)$$

At $\varkappa = 1$ the result for an isotropic plate is recovered, [2].

For a plate strip, $\alpha \to \infty$ and

$$p = 2\sqrt{\varkappa}(\pi+2). \quad (9.14)$$

Influence of the reinforcement arrangement on the load carrying capacity is illustrated in Fig. 6, where also a comparison of (9.13) with the yield line theory solutions is given.

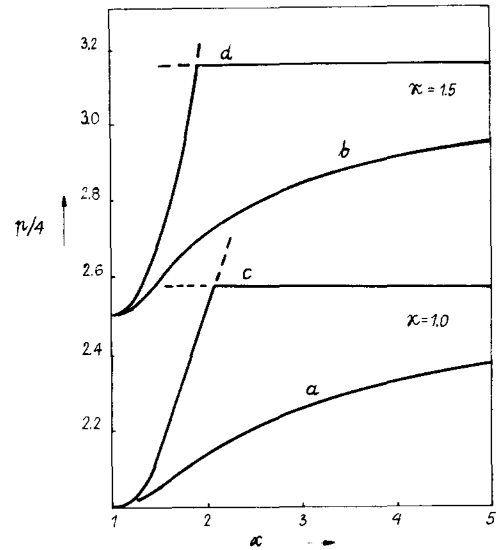

FIG. 6. Collapse load for orthotropic plate; exact: (a) $\varkappa = 1$, (b) $\varkappa = 1, 5$; yield line theory: (c) $\varkappa = 1$, (d) $\varkappa = 1, 5$.

(e) *Clamped circular plate loaded at two points, lower bound solution*

In plates loaded by systems of concentrated forces discontinuous stress fields develop. For the case shown in Fig. 7, CB is a discontinuity line on transition from the hyperbolic region AC to the zone of homogeneous stress.

In the hyperbolic zone $m_r, m_{r\theta}, m_\theta$ are as given by (8.7) and (8.2a) respectively, whereas in OCB

$$m_x = 1, \qquad m_y = \varkappa, \qquad m_{xy} = 0. \quad (9.15)$$

This homogeneous stress field satisfies on OB the continuity requirements $[m_\theta] = [m_{r\theta}] = 0$. The radial moment and shear have the discontinuity

$$[m_r] = r[q_r] = -2\varkappa/(\sin^2\theta + \varkappa\cos^2\theta). \quad (9.16)$$

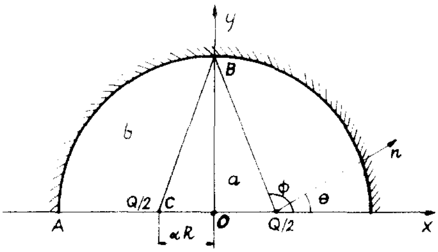

FIG. 7. Two point loaded circular plate: discontinuous field (a) homogeneous bending, (b) hyperbolic regime.

As it was pointed out earlier, the zone of homogeneous bending carry no load. Thus the load carrying capacity of the plate is found from (8.9) to be

$$p = 8\sqrt{\varkappa}[\pi - \mathrm{ctg}^{-1}\,\alpha\sqrt{\varkappa}]. \tag{9.17}$$

The method can readily be applied to other examples. It is to be noted that for the yield condition (3.8) no essential difference exists between the procedure of collapse load evaluation for isotropic and orthotropic plates. The respective relations for collapse loads differ however, since the characteristics are no longer lines of principal moments. Kinematical conditions on a clamped boundary require a special study.

REFERENCES

[1] K. W. JOHANSEN, *Brudlinieteorier*. Teknisk Forlag (1942).
[2] A. SAWCZUK and P. C. HODGE, JR., Limit analysis and yield line theory. *J. appl. Mech.* (1968). Paper 68APM-I.
[3] CH. MASSONNET, M. SAVE, *Calcul Plastique des Constructions*, Vol. 2, ASBL (1963).
[4] A. SAWCZUK, TH. JAEGER, *Grenztragfähigkeits-Theorie der Platten*. Springer (1963).
[5] M. SAVE, A consistent limit-analysis theory for reinforced concrete slabs. *Mag. concr. Res.* **19**, 3–12 (1967).
[6] M. P. NIELSEN, Limit analysis of reinforced concrete slabs. *Acta Polytech. Scand. Ci* 26 (1964).
[7] C. T. MORLEY, On the yield criterion of an orthogonally reinforced concrete slab element. *J. Mech. Phys. Solids*, **14**, 33–47 (1966).
[8] K. O. KEMP, The yield criterion for orthotropically reinforced concrete slabs. *Int. J. Mech. Sci.*, **7**, 737–746 (1965).
[9] W. SCHUMANN, On limit analysis of plates. *Q. appl. Math.* **16**, 61–71 (1958).
[10] W. OLSZAK, Probleme der Grenzlasttheorie der orthotropen Platten. *Acta Tech. Hung.* **16**, 3–37 (1956).
[11] H. C. HOPKINS, On the plastic theory of plates. *Proc. R. Soc.* **A241**, 153–179 (1957).
[12] CH. MASSONNET, Complete solutions describing the limit state of reinforced concrete slabs. *Mag. concr. Res.* **19**, 13–22 (1967).
[13] R. H. WOOD, *Plastic and Elastic Analysis of Slabs and Plates*. Thames & Hudson (1961).

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Абстракт—Представляется метод определения полных решений граничного состояния для ортотропных пластинок. Показано существование гиперболических режимов напряжений для условия текучести, соответствующего железобетонным плитам. Наличие гипероболических областей вызывает разницы между точным решением и решением теории линий перелома. Указываются разницы между формулами изотропной и ортотропной пластинки для пределных нагрузок. Рассматриваются примеры при разных граничных условиях.